

Time Crystals

Symmetry and its spontaneous breaking are at the heart of our modern understanding of the physical world.

Spontaneous breaking of spatial translation symmetry is, of course, very common (as common as crystals).

It is easy to construct mean field theories for that phenomenon:

$$V(\phi) = -\mu^2 (\nabla\phi)^2 + \lambda ((\nabla\phi)^2)^2$$

$$V(\phi_\alpha) = Q^{\alpha\beta} \partial_i \phi_\alpha \partial^i \phi_\beta + \Lambda^{\alpha\beta\gamma\delta} \partial_i \phi_\alpha \partial^i \phi_\beta \partial_j \phi_\gamma \partial^j \phi_\delta$$

Inspired by special relativity, or simply by analogy, it is natural to consider the possibility of spontaneous breaking of *time* translation symmetry.

This turns out to bring in some surprising novelties.

Classical Time Crystals

I. At first sight, looking at the Hamilton equations, the idea seems like a non-starter:

$$\dot{p}_j = - \frac{\partial H}{\partial q^j}$$
$$\dot{q}^j = \frac{\partial H}{\partial p_j}$$

Minimum energy \Rightarrow nothing moves (it seems)

2. On the other hand, the Lagrangian analogue of our earlier, “crystalline” potential V suggests something different:

$$L = \frac{1}{4}\dot{\phi}^4 - \frac{\kappa}{2}\dot{\phi}^2$$

$$E = \frac{3}{4}\dot{\phi}^4 - \frac{\kappa}{2}\dot{\phi}^2$$

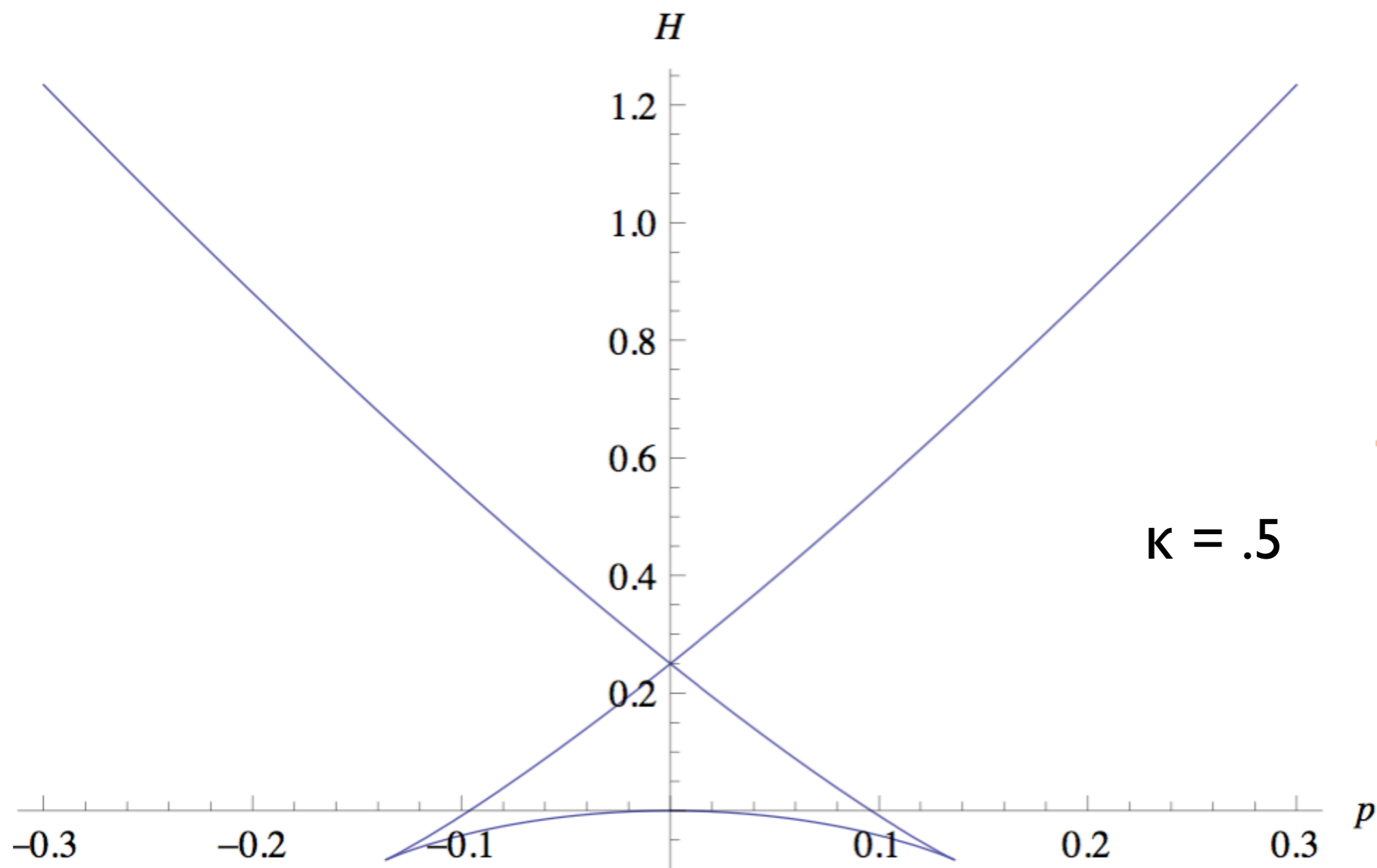
\Rightarrow at energy minimum: $\dot{\phi}^2 = \frac{\kappa}{3}$

3. What has happened here?

The point is that our innocuous-looking L leads to a cuspy $H(p)$:

$$E = \frac{3}{4} \dot{\phi}^4 - \frac{\kappa}{2} \dot{\phi}^2$$

$$p = \dot{\phi}^3 - \kappa \dot{\phi}$$



The energy minima occur at the cusps, so they needn't - and don't! - have horizontal slopes.

4. When we add a potential, the energy becomes

$$E = \frac{3}{4}\dot{\phi}^4 - \frac{\kappa}{2}\dot{\phi}^2 + V(\phi)$$

If we choose initial conditions to minimize both the kinetic part *and* V , we've got a problem!

5. More generally, the equation of motion

$$\ddot{\phi} \left(\dot{\phi}^2 - \frac{\kappa}{3} \right) = - \frac{\delta V}{\delta \phi}$$

breaks down at

$$\dot{\phi}^2 = \frac{\kappa}{3}$$

unless the right-hand side also vanishes there.

6. On the other hand, we are safe for

$$E \geq -\frac{1}{12}\kappa^2 + V_{\max}.$$

7. The connection between “breakdown” of the equation of motion and breaking of time translation symmetry (i.e., motion in the ground state) is general:

$$E = \frac{\partial L}{\partial \dot{\phi}_j} \dot{\phi}_j - L$$

$$\frac{\partial E}{\partial \dot{\phi}_k} = \frac{\partial^2 L}{\partial \dot{\phi}_j \partial \dot{\phi}_k} \dot{\phi}_j$$

$$\left(\frac{\partial L}{\partial \dot{\phi}_k} \right) \cdot = \frac{\partial^2 L}{\partial \dot{\phi}_k \partial \dot{\phi}_j} \ddot{\phi}_j + \dots \quad (\text{eqn. of motion})$$

So if the energy minimization condition $\frac{\partial E}{\partial \dot{\phi}_k} = 0$ has a non-trivial solution $\dot{\phi}_j^0$, then the eqn. of motion does not constrain the component of acceleration $\propto \dot{\phi}_j^0$

8. It is understandable that it shouldn't be too easy to get spontaneous motion.

If we have motion associated with the minimum energy, then we must have a whole curve of states with that minimum energy; and that degeneracy requires special conditions.

Are there non-trivial models?

Yes there are, two kinds:

Fine-tuned.

Natural, connected with broken
symmetry.

9. Lagrangians of the form

$$L = f\dot{\phi}^4 + g\dot{\phi}^2 + h$$

for **functions** f, g, h lead to energies of the form

$$\begin{aligned} E &= 3f\dot{\phi}^4 + g\dot{\phi}^2 - h \\ &= 3f\left(\dot{\phi}^2 + \frac{g}{6f}\right)^2 - \frac{g^2}{12f} - h. \end{aligned}$$

So if $\frac{g^2}{12f} + h = \text{const.}$

the energy is indeed minimized on a curve, for any $f > 0, g < 0$).

The energy will be minimized for

$$\dot{\phi}_0^2 = -\frac{g}{6f}$$

In this way we see that any orbit with a velocity that does not change sign can be realized, in many ways, as the stable minimum of an appropriate, reasonably simple Lagrangian.

10. Semi-classical quantization can avoid the singular zone, even with (small) potentials.

11. We can also get *natural* models by having the trajectories move along the orbit of a spontaneously broken symmetry. This is an interesting and promising direction.

12. Special cases: “Universal” Q ball, current-carrying ground state, traveling density waves, ...

Quantum Time Crystals

I. Quantum mechanically, one has a similar (apparent) “no-go”:

$$\langle \psi | \dot{\mathcal{O}} | \psi \rangle = -i \langle \psi | [H, \mathcal{O}] | \psi \rangle \stackrel{?}{=} 0$$

would appear to vanish for the ground state, for any potential order parameter of motion.

2. Also: A system with spontaneous τ breaking appears perilously close to being a perpetual motion machine.

3. Also: By what criterion are we going to pick out “ground states” that break τ (since minimizing $\langle H \rangle$ won't do it)?

And Yet ...

1. In the right circumstances, supercurrents will flow forever in the ground state.

This is suggestive; though if the current is constant, time translation symmetry τ is not broken. (Time-reversal T of course is).

2. We can capture an important part of the essence of the matter by considering a very simple quantum mechanical model, to wit a charged particle on a ring threaded by magnetic flux.

$$L = \frac{1}{2}\dot{\phi}^2 + \alpha\dot{\phi}$$

$$H = \frac{1}{2}(\pi_{\phi} - \alpha)^2$$

$$E_l = \frac{1}{2}(l - \alpha)^2$$

$$\langle l | \dot{\phi} | l \rangle = l - \alpha$$

Thus if α is not an integer*, the ground state will support non-zero $\langle l_0 | \dot{\phi} | l_0 \rangle$

This is a direct consequence of the quantization of (canonical) angular momentum.

Since $\dot{\phi}$ is neither the time derivative of a well-defined operator, nor the commutator of the Hamiltonian with one, there is no contradiction.

*(The half-odd integer case requires special consideration.)

Time reversal T is generally broken
(intrinsically*, usually), but not time-
translation τ .

3. What might appear to be a special difficulty with breaking τ , because of its connection to the Hamiltonian, actually arises for all cases of spontaneous symmetry breaking. Consider a complex order parameter that acquires a non-zero vacuum expectation value, which we can take to be real:

$$\langle 0 | \Phi | 0 \rangle = v \neq 0$$

We will also have alternative energetically degenerate, orthogonal** states with arbitrary values of the phase

$$\langle \sigma | \Phi | \sigma \rangle = v e^{i\sigma}$$

The superposition

$$|\Omega\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\sigma |\sigma\rangle$$

is therefore energetically degenerate, and more symmetric:

$$\langle \Omega | \Phi | \Omega \rangle = 0$$

****The different σ states are orthogonal in the limit of infinite volume (or, more generally, an infinite number of correlated degrees of freedom), because we must multiply smaller-than-unity overlap factors infinitely many times.**

$$\Psi_{\sigma}(x_1, \dots, x_N) \approx \prod_{j=1}^N \psi_{\sigma}(x_j)$$

$$\langle \Psi_{\sigma'} | \Psi_{\sigma} \rangle \approx \prod_{j=1}^N \langle \psi_{\sigma'} | \psi_{\sigma} \rangle = (f_{\sigma'\sigma})^N \rightarrow 0 \quad (\sigma' \neq \sigma)$$

The comment (**) is also the key to why we prefer the σ states to Ω . No finite product of local observables connects different σ states, so any world of observation will correspond to a single σ state:

$$\langle \Psi_{\sigma'} | \mathcal{O}_1(x_a, x_b) \mathcal{O}_2(x_c, x_d, x_e) \dots | \Psi_{\sigma} \rangle \quad \propto \quad (f_{\sigma'\sigma})^{N-\text{finite}} \quad \rightarrow \quad 0$$

$(\sigma' \neq \sigma)$

Thus the physical criterion that identifies useful “ground states” is not simply energy, but also observability.

The infinite volume (or infinite DOF) limit is crucial here; finite systems cannot exhibit spontaneous symmetry breaking.

Model

Existence Proof

With this background, we are ready to construct a simple model of τ violation.

We take an infinite number ($N \rightarrow \infty$) of copies of our ring-particles, with an identical attractive δ -function interaction between each pair.

$$H = \sum_{j=1}^N \frac{1}{2} (\pi_j - \alpha)^2 - \frac{\lambda}{N-1} \sum_{j \neq k} \delta(\phi_j - \phi_k)$$

The particles will want to be in the same place, and (for noninteger α) they will want to move.

So we can expect to get a moving lump.
Such a dynamical configuration will violate τ ,
giving us a time crystal.

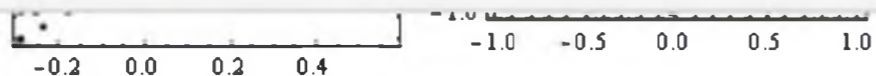
Consider first $\alpha = 0$.

We take a product wave-function *ansatz*, and saturate with it (“mean field” approximation).

(Operationally, we define an effective $V_{\text{eff.}}(\Phi)$ with

$\langle \psi(\Phi) | V_{\text{eff.}}(\Phi) | \psi(\Phi) \rangle = \langle \Psi(\Phi_j) | V(\Phi_j) | \Psi(\Phi_j) \rangle$
by integrating over $N-1$ variables.)

In this way we arrive at a non-linear Schrödinger equation. Its localized “soliton” solution, constrained to be periodic, can be expressed in elliptic functions:



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Solution of Nahm equations:

```
In[1]:= T1[z_] = -Sqrt[m] JacobiSN[z, m] {{0, 1, 0}, {1, 0, 1}, {0, 1, 0}} / Sqrt[2];
T2[z_] = I Sqrt[m] JacobiCN[z, m] {{0, I, 0}, {-I, 0, I}, {0, -I, 0}} / Sqrt[2];
T3[z_] = I JacobiDN[z, m] {{1, 0, 0}, {0, 0, 0}, {0, 0, -1}};
```

Check that the solutions fulfill the Nahm equations:

```
In[2]:= T1'[z] - I (T2[z] . T3[z] - T3[z] . T2[z])
```

```
Out[2]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
In[3]:= T2'[z] - I (T3[z] . T1[z] - T1[z] . T3[z])
```

```
Out[3]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
In[4]:= T3'[z] - I (T1[z] . T2[z] - T2[z] . T1[z])
```

```
Out[4]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
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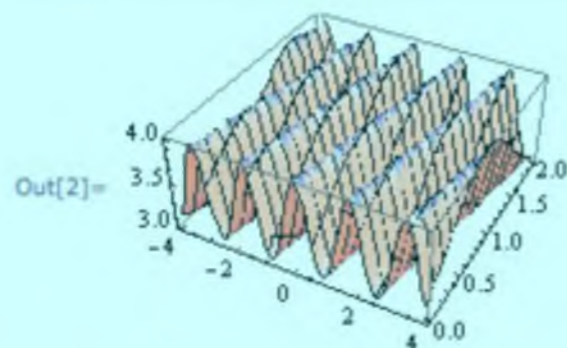
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Periodic solution of the nonlinear Schrödinger equation $i\partial_t \psi(x, t) = -\partial_{xx} \psi(x, t) - 2|\psi(x, t)|^2 \psi(x, t) = 0$:

```
In[1]:=  $\psi[x_, t_] := r \text{Exp}[I (p x - (p^2 - (2 - k^2) x^2) t)] \text{JacobiDN}[r x - 2 p r t, k^2]$ 
```

Plot the solution:

```
In[2]:= Plot3D[Evaluate[Abs[ $\psi[x, t]$ ]2 /. {r -> 2, p -> 1/3, k -> 1/2}], {x, -4, 4}, {t, 0, 2}]
```



.....

Parametrize a lemniscate by arc length [more info]:

```
In[1]:= lemniscate[s_] := {JacobiCN[Sqrt[2] s, 1/2] JacobiDN[Sqrt[2] s, 1/2],
  JacobiCN[Sqrt[2] s, 1/2] JacobiSN[Sqrt[2] s, 1/2] / Sqrt[2]}
```

Show arc length parametrization and classical parametrization:

From Faraday's law, we can expect that turning on the flux, i.e. cranking up α , torques the lump in a simple way.

In fact we both boost and “gauge-transform”, to get interacting versions of our earlier l states.

Specifically: We can solve

$$i \frac{\partial \psi_l}{\partial t} = \frac{1}{2} (-i \partial_\phi - \alpha)^2 \psi_l - \lambda |\psi_l|^2 \psi_l$$

with

$$\psi_l(\phi, t) = e^{-il\phi} \tilde{\psi}(\phi + (l + \alpha)t, t)$$

and

$$i \frac{\partial \tilde{\psi}}{\partial t} = \frac{1}{2} (-i \partial_\phi)^2 \tilde{\psi} - \lambda |\tilde{\psi}|^2 \tilde{\psi} + \frac{(l + \alpha)^2}{2} \tilde{\psi}$$

The minimal energy solution occurs for the minimum value of $(l + \alpha)$. If that quantity is not an integer, the solution is a moving lump. τ is then broken.

Discussion

Were we *literally* talking about charged particles, we'd have to worry about coupling to the (dynamical) electromagnetic field, leading to radiation.

Formal mechanism: “Long-wavelength” photons couple to all the ϕ_j . The time-evolution operator, which exponentiates this interaction, is non-local.

So we should expect relaxation to the Ω state.

We can ameliorate this by using multipoles, or by putting a gap in the photon spectrum, e.g. by working in a cavity, insuring slow and possibly inconsequential relaxation.

On more philosophical and speculative notes:

We've been discussing the spontaneous emergence of clocks.

More complex systems of this kind, taking excursions in a large, structured Hilbert space, could be quantum computers capable of dodging the heat death of the universe for a very long time.

Imaginary Time Crystals

We represent the partition function as an integral over all configurations periodic in Space-iTime, with iTime period $\beta = 1/T$, weighted by e^{-S} .

Thus the partition function for a d -spatial dimension system is a weighted sum over $d+1$ dimensional Euclidean field configurations.

If the ground state of the $d+1$ dimensional theory is crystalline, the partition function will be dominated by iTime crystals.

The crystal structure will fit without distortion if and only if the iTIME lattice period divides $\beta = 1/T$ evenly.

Thus a signature for iTIME crystalline states is some approximately periodic behavior in $1/T$, especially for small T .

Directions

Many questions and possibilities for development arise:

Directed 0-point motion, more generally?

Concrete, practical realizations?

$T \neq 0$ in real time?

Classification of space-time and space-iTime crystals.

All the usual SSB questions: excitation spectrum, phase transitions (critical dimensions), defects, ...

END

[slide dump follows]

Sombrero Doble

1. To support motion in the ground state, we want constant energy along its orbit. Orbits of constant energy are typically associated with symmetry, so it is natural to look for realizations in systems with symmetry.

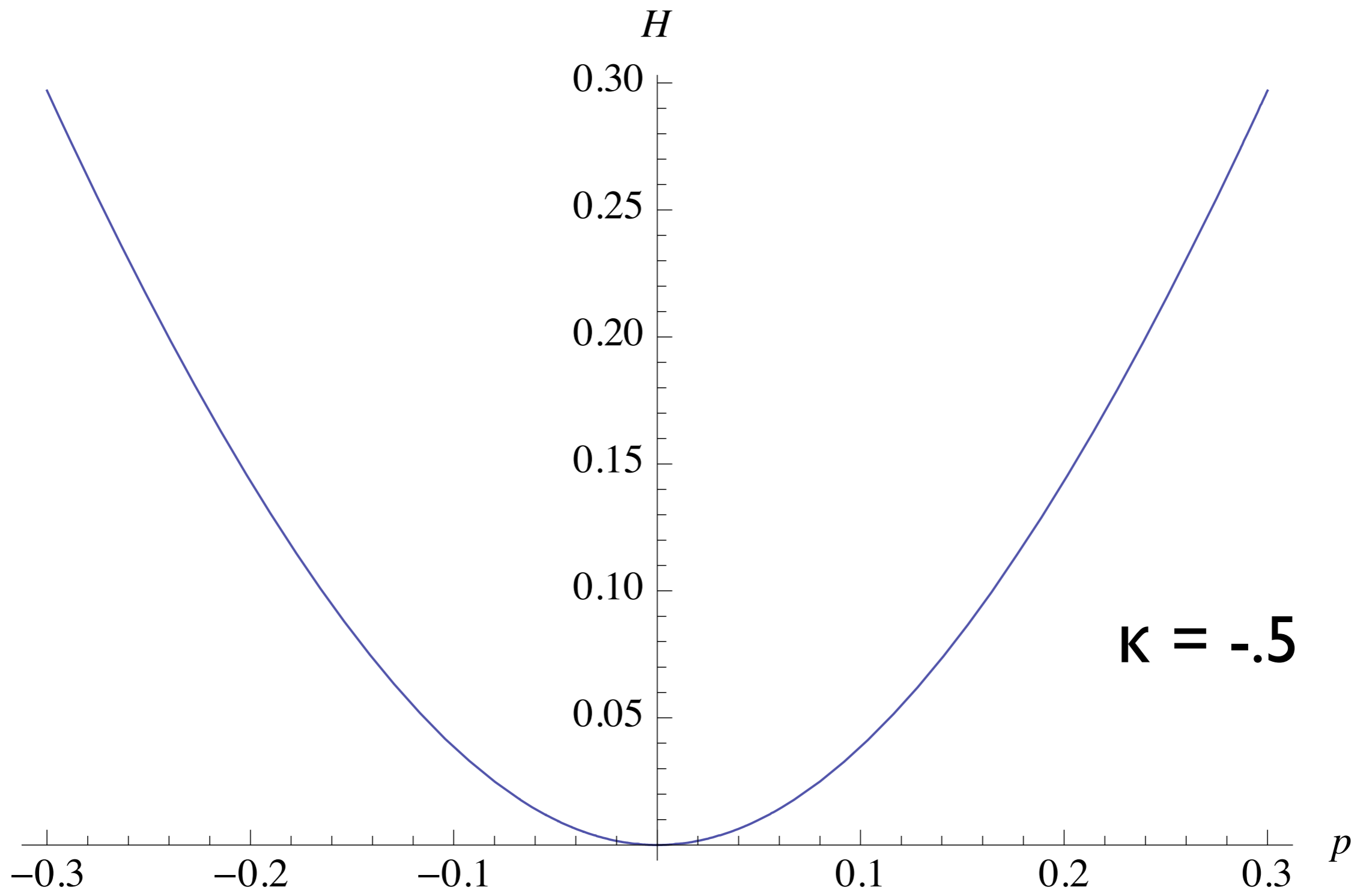
2. The simplest example structure, conceptually, is a double Mexican hat, or sombrero doble. We envision a radial field ρ with a sombrero potential governing its magnitude; and an associated angular field ϕ of the type we've been discussing, with a sombrero kinetic Lagrangian.

3. The polynomial building blocks for invariant terms include:

$$\begin{aligned}\dot{\psi}_1^2 + \dot{\psi}_2^2 &= \dot{\rho}^2 + \rho^2 \dot{\phi}^2 \\ \psi_1 \dot{\psi}_2 - \psi_2 \dot{\psi}_1 &= \rho^2 \dot{\phi} \\ (\psi_1^2 + \psi_2^2) \cdot &= 2\rho \dot{\rho} \\ \psi_1^2 + \psi_2^2 &= \rho^2\end{aligned}$$

4. The sort of structure we want will appear if we have the square of the second term appearing with a negative coefficient, controlled by the square of the first term. There is considerable latitude, given that core structure.

As K changes sign, the qualitative form of the energy function changes. It is the unfolding of a mathematical “catastrophe”.



Dynamics (and
“Soundness”) of the fgh
Model

1. As a sort of existence proof / sanity check, I'd like to demonstrate that the fgh model has sensible dynamics, and a good initial value problem in general (not just for $f, g, h = \text{const.}$).

2. The energy is

$$E = f \left(\dot{\phi}^2 + \frac{g}{6f} \right)^2$$

and so

$$\dot{\phi}^2 + \frac{g}{6f} = \pm \sqrt{\frac{E}{f}}$$

(Recall $f > 0$, $g < 0$; and obviously $E \geq 0$.) This is the equation for motion in an effective potential.

$$\dot{\phi}^2 + \frac{g}{6f} = \pm \sqrt{\frac{E}{f}}$$

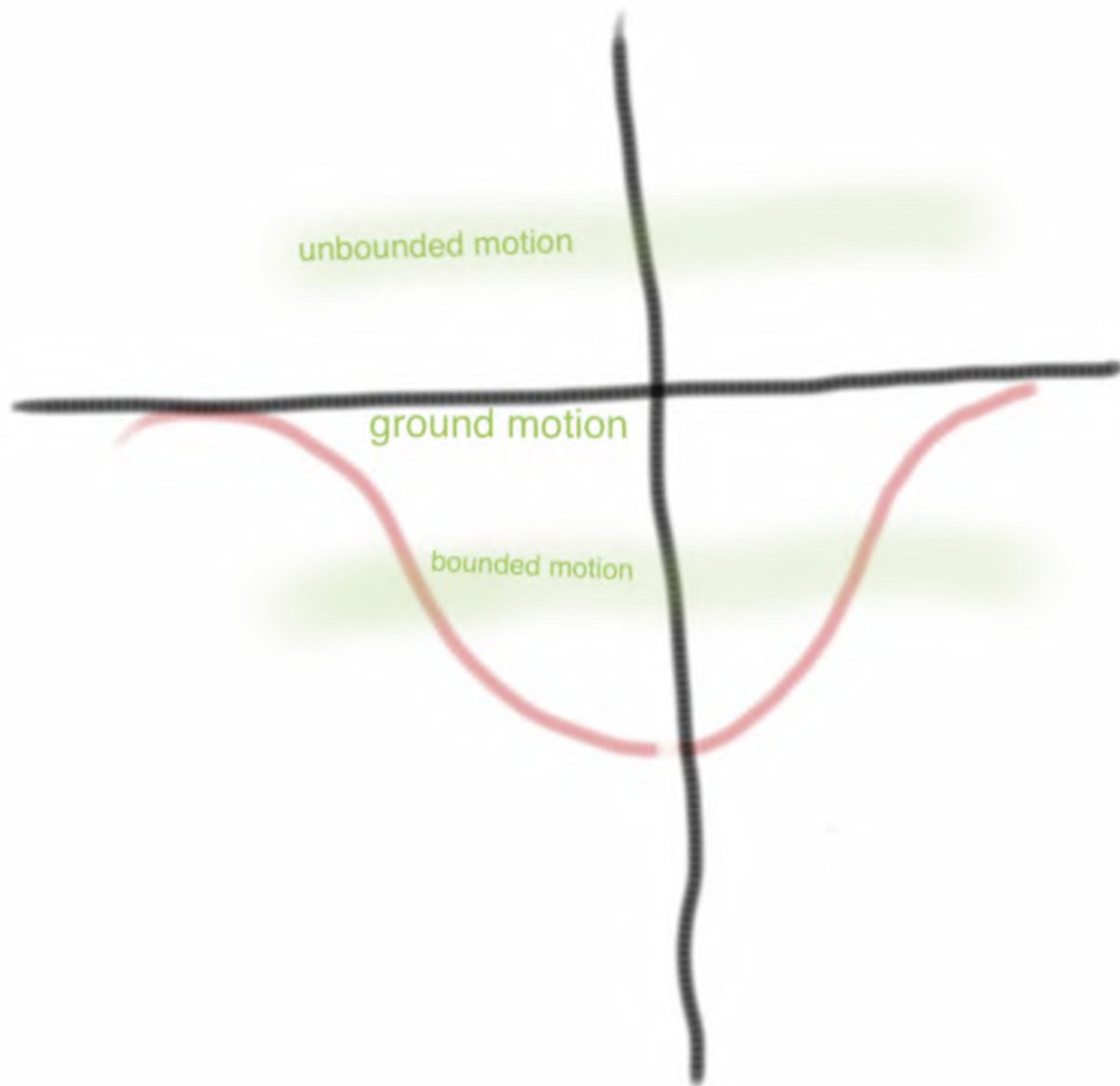
Thus we can use our experience in mechanics to anticipate the motion.

3. Assume for simplicity $f = 1$. Then we have motion in the (negative) effective potential

$$V_{\text{eff.}} = \frac{g}{6f}$$

with the *effective*, “fictitious” energy

$$\begin{aligned} E_{\text{eff.}} &= \pm \sqrt{E} \\ E &= E_{\text{eff.}}^2 \end{aligned}$$



4. It's entertaining, I think, that the bounded motions have higher energy than many unbounded motions.